

# Schwarzschild Black Holes in Various Dimensions from Matrix Theory

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## Abstract

In a recent paper it was shown that the properties of Schwarzschild black holes in 8 dimensions are correctly described up to factors of order unity by Matrix theory compactified on  $T^3$ . Here we consider compactifications on tori of general dimension  $d$ . Although in general little is known about the relevant  $d + 1$  dimensional theories on the dual tori, there are hints from their application to near-extreme parallel Dirichlet  $d$ -branes. Using these hints we get the correct mass–entropy scaling for Schwarzschild black holes in  $(11 - d)$  dimensions.

September 1997

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# 1 Introduction

In a recent paper [1] Schwarzschild black holes in 8 dimensions were studied from the point of view of Matrix theory [2] compactified on  $T^3$ . In this particular case enough is known about the relevant SYM theory on the dual torus to derive the properties of black holes, including the mass–entropy relation and the physical size, up to numerical factors of order unity. An important ingredient in this work was the knowledge of the equation of state. Certain special features of the  $3 + 1$  dimensional SYM theory with 16 supercharges, such as its conformal invariance, require the equation of state to be of the form,

$$E \sim N^2 V_d T^4, \quad S \sim N^2 V_d T^3, \quad (1)$$

where  $V_d$  is the volume of the dual torus. This equation of state is supported by the form of the near-extremal entropy of the self-dual 3-brane found in [3, 4].

Here we consider generalization of [1] to compactification of Matrix theory on  $T^d$ . For general  $d$  little is known about the relevant large  $N$  SYM theory from first principles. However, the same theory is supposed to describe  $N$  coincident Dirichlet  $d$ -branes [5]. Thus, we will assume that the equation of state for this theory can be read off from the near-extremal entropy of the RR charged classical  $d$ -brane solution that was found in [4]. Then we will see that the same strategy that has worked for the  $D = 8$  Schwarzschild black holes [1] continues to work in other dimensions, and we find the correct scaling of mass vs. entropy.

Let us review the strategy. Matrix theory is best thought of as the Discretized Light-Cone Quantization (DLCQ) of M-theory [6], i.e. compactification on a light-like circle of radius  $R$ . Accordingly, the longitudinal momentum  $P_- = P^+$  is quantized in integer multiples of  $1/R$ ,

$$P_- = \frac{N}{R}. \quad (2)$$

We further compactify  $d$  transverse coordinates on a  $d$ -dimensional torus. For simplicity, we consider this torus to be “square” with equal circumferences given by  $L$ .

The Matrix theory conjecture is that the sector of the theory with given value of  $N$  is exactly described by  $U(N)$  SYM theory in  $d + 1$  dimensions with 16 real supercharges. This theory lives on a dual torus with circumferences [7, 8]

$$\Sigma \sim \frac{l_{11}^3}{RL}. \quad (3)$$

For physical applications  $N$  has to be taken sufficiently large to achieve desired “resolution” of a given system, i.e. to capture the important degrees of freedom. In particular, a black hole of size  $R_s$  has to be boosted to sufficient momentum that its longitudinal dimension fits within the circle of radius  $R$ . In [1] it was shown that this requires  $N$  to be at least of the order of the black hole entropy,  $N \sim S$ . This is the regime where we will be working.

Determination of the mass–entropy relation for the black hole goes as follows. We start with the relation between the Matrix theory energy and entropy (the equation of state) for

given  $N$ ,

$$E = E(N, S) . \quad (4)$$

Next we observe that the Matrix theory hamiltonian is identified with the DLCQ energy according to

$$E = \frac{M^2}{P_-} = \frac{M^2 R}{N} . \quad (5)$$

Thus, we find

$$M^2 = \frac{N}{R} E(N, S) . \quad (6)$$

Choosing  $N \sim S$ , we have

$$M^2 \sim \frac{S}{R} E(S, S) . \quad (7)$$

Note that the Matrix hamiltonian is explicitly proportional to  $R$ , so that  $R$  cancels in (7) leaving a relation between mass and entropy.

In [1] it was argued that there is a sector of the 3 + 1 dimensional SYM theory where the equation of state (1) indeed holds down to entropies of order  $N$ , where the temperature is very low,

$$T \sim \frac{1}{N^{1/3} \Sigma} . \quad (8)$$

This sector may be thought of as a single 3-brane, each of whose sides is wrapped  $N^{1/3}$  times over the torus. Using the equation of state (1) for  $d = 3$ , and applying the above strategy, it was found in [1] that

$$S \sim M^{6/5} G_8^{1/5} , \quad (9)$$

which is correct for  $D = 8$  black holes.

## 2 Schwarzschild Black Holes In $D \neq 8$

Analysis of Schwarzschild black holes in dimensions  $D \neq 8$  is hampered by the lack of understanding of the relevant SYM theory in  $d + 1$  dimensions ( $d = 11 - D$ ). Nevertheless, we can get useful information on its equation of state by studying near-extremal RR charged  $d$ -branes, as was done in [4]. We will show that, if we use such an equation of state, then the mass–entropy relation works out correctly for Matrix theory Schwarzschild black holes in all dimensions. We should emphasize that we do not have a first principles derivation of these equations of state and are puzzled by some of their strange features, such as the negative specific heat for  $d > 5$ . Nevertheless, it is very interesting that the approach to Schwarzschild black holes that has worked for  $D = 8$  continues to work in other dimensions.

First we demonstrate a few examples, and then work out the general formula. For  $d = 1$  we conjecture the following equation of state,

$$S \sim N^{3/2} \Sigma T^2 g^{-1} , \quad E \sim N^{3/2} \Sigma T^3 g^{-1} . \quad (10)$$

The scalings with respect to  $N$ ,  $\Sigma$  and  $T$  are exactly as for near-extremal RR charged strings [4]. The only other quantity in the SYM theory that can be used to make the formulae dimensionally correct is the coupling  $g$ , and we have inserted it appropriately. The power  $N^{3/2}$  seems surprising, but if we follow the usual large  $N$  logic and introduce  $e = g\sqrt{N}$  then (10) becomes<sup>2</sup>

$$S \sim N^2 \Sigma T^2 e^{-1}, \quad E \sim N^2 \Sigma T^3 e^{-1}, \quad (11)$$

consistent with there being  $O(N^2)$  degrees of freedom.

The equation of state (10) is like that of a superconformal theory in  $2 + 1$  dimensions. Thus, we conjecture that the  $1 + 1$  dimensional large  $N$  SYM theory with 16 supercharges develops an extra dimension of size  $1/g$  (obviously, this dimension is macroscopic only for weak coupling). The coupling constant is given by [10]

$$g^2 \sim \frac{R^2}{L l_{11}^3}. \quad (12)$$

Using (3) we find that the dimensionless coupling  $g\Sigma$  is independent of  $R$ . Just as for  $d = 3$  our strategy is to work with  $N \sim S$ . From (12) and (3) we find

$$T \sim N^{-1/4} R L^{1/4} l_{11}^{-9/4}. \quad (13)$$

Applying the strategy reviewed in the introduction, we find that the  $R$  dependence cancels out, which means that the equation of state (10) gives results consistent with the boost invariance. We arrive at the following mass–entropy relation,

$$S \sim M^{8/7} G_{10}^{1/7}, \quad (14)$$

which is indeed correct for  $D = 10$  Schwarzschild black holes.

It is important to determine the range of validity of the equation of state (10).<sup>3</sup> We expect (10) to hold for temperatures no greater than  $T_{max}$ . For  $T > T_{max}$  the theory should become approximately free, with the equation of state,

$$S \sim N^2 \Sigma T, \quad E \sim N^2 \Sigma T^2. \quad (15)$$

This equation of state was used in [9] to match the entropy of D1-branes sufficiently far from extremality. Comparing with (10), we see that the transition temperature is  $T_{max} \sim N^{1/2} g$ . This is exactly where we expect it to be, since here the large  $N$  dimensionless coupling parameter,

$$\kappa = \frac{g^2 N}{T^2}, \quad (16)$$

is of order one. For  $T \gg T_{max}$ ,  $\kappa \ll 1$  and the theory is close to being free. The temperature (13) is far below  $T_{max}$  for large  $N$ . Thus, we are justified in using (10) as the equation of

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<sup>2</sup>We are grateful to A. Polyakov for a useful discussion on this.

<sup>3</sup>We are grateful to E. Witten for illuminating remarks on this issue.

state for  $S \sim N$ . For  $S < N$  we believe that the equation of state assumes yet another form, which can be deduced from the expectation that  $M^2$  computed from (6) is independent of  $N$  and satisfies (14). Thus, for  $S < N$  we are led to conjecture the equation of state

$$E(N, S) \sim \frac{1}{N} S^{7/4} g^{1/2} \Sigma^{-1/2} . \quad (17)$$

Another instructive special case is  $d = 4$ . Here we conjecture the following equation of state,

$$S \sim N^3 \Sigma^4 T^5 g^2 , \quad E \sim N^3 \Sigma^4 T^6 g^2 . \quad (18)$$

The scalings with respect to  $N$ ,  $\Sigma$  and  $T$  are exactly as for near-extremal RR charged 4-branes [4], and we have added a power of the SYM coupling  $g$  to make the dimensions correct. In terms of the coupling  $e$  kept finite in the large  $N$  limit, we have

$$S \sim N^2 \Sigma^4 T^5 e^2 , \quad E \sim N^2 \Sigma^4 T^6 e^2 . \quad (19)$$

The fact that in our conjectured equations of state  $S$  and  $E$  scale as  $N^2$  for fixed  $T$ ,  $\Sigma$  and  $e$  is true for general dimension  $d$ .

The equation of state (18) is consistent with the idea that the theory develops an extra dimension of length  $g^2$  [11, 10]. Thus, it behaves like a superconformal theory in  $5 + 1$  dimensions. We believe that (18) is valid down to entropies of order  $N$ . Using [10]

$$g^2 \sim \frac{l_{11}^6}{RL^4} \quad (20)$$

and (3) we find

$$T \sim N^{-2/5} RL^{8/5} l_{11}^{-18/5} . \quad (21)$$

Proceeding in the usual way we arrive at

$$S \sim M^{5/4} G_7^{1/4} , \quad (22)$$

which is correct for  $D = 7$  Schwarzschild black holes.

Now consider a general case of Matrix theory compactified on  $T^d$ . For longitudinal momentum  $N$  we are dealing with  $U(N)$  SYM theory in  $d + 1$  dimensions. The same theory describes  $N$  coincident Dirichlet  $d$ -branes. For large  $N$ , we can use the near-extremal RR-charged  $d$ -brane solution to read off the relation between the entropy and the energy for such a theory [4],

$$S \sim \sqrt{N} E^\lambda \Sigma^{d(1-\lambda)} g^a , \quad 2\lambda = \frac{D-2}{D-4} . \quad (23)$$

The power of the gauge coupling is determined by dimensional analysis,

$$a = \frac{8-D}{D-4} . \quad (24)$$

We also need the expression for the coupling [10],

$$g^2 \sim \frac{l_{11}^{3d-6} R^{3-d}}{L^d} . \quad (25)$$

Now consider this theory for entropy of order  $N$ .<sup>4</sup> We find from (23) that

$$E \sim RS^{\frac{D-4}{D-2}} \left( \frac{L^d}{l_{11}^9} \right)^{\frac{2}{D-2}} . \quad (26)$$

Repeating the by now familiar steps, we find

$$S \sim M^{\frac{D-2}{D-3}} G_D^{\frac{1}{D-3}} , \quad (27)$$

where  $G_D = l_{11}^9/L^d$ . Thus, the scaling of the entropy vs. the mass works out correctly for  $4 \leq D \leq 11$ . Note, however, that the equation of state (23) becomes singular for  $D = 4$  ( $d = 7$ ).

The cases  $d = 5$  and  $d = 6$  are also special in that they are best described in a microcanonical ensemble. For  $d = 5$  the entropy (23) is characteristic of dynamical strings [12]. Interpreting the temperature following from (23) as the Hagedorn temperature of the effective strings, we find that their tension is

$$T_{eff} \sim \frac{1}{Ng^2} . \quad (28)$$

This is consistent with the strings being instantons of fractional instanton number  $1/N$ .

For  $d = 6$ , (23) has the form characteristic of dynamical 3-branes [13]. However, the negative specific heat encountered here suggests that compactification of Matrix theory on  $T^6$  may involve radically new physics. Indeed, there are other indications for this.<sup>5</sup>

## Acknowledgments

We are grateful to C. Callan, A. Polyakov and E. Witten for discussions on the equations of state and to N. Seiberg for discussions of compactification on  $T^6$ . We also thank T. Banks and A. Rajaraman for discussions of compactification on  $T^4$ . The work of I.R.K was supported in part by the DOE grant DE-FG02-91ER40671, the NSF Presidential Young Investigator Award PHY-9157482, and the James S. McDonnell Foundation grant No. 91-48. L.S. is grateful to IAS for hospitality. L.S. is a Raymond and Beverly Sackler Fellow at IAS and is supported in part by NSF grants PHY-9513835 and PHY-9219345.

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<sup>4</sup> For  $d < 3$  considerations of the range of validity, such as those given earlier for  $d = 1$ , show that (23) is valid only up to some maximum entropy,  $S_{max}$ . For  $S > S_{max}$  we instead find the equation of state of a free field theory.  $S_{max}$  turns out to be much greater than  $N$ , so that our approach is self-consistent.

<sup>5</sup>N. Seiberg, private communication.

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